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Abstract
An algebraic criterion for the vanishing of the beta function for renormalizable quantum field theories is presented. Use is made of the descent equations following from the Wess–Zumino consistency condition. In some cases, these equations relate the fully quantized action to a local gauge invariant polynomial. The vanishing of the anomalous dimension of this polynomial enables us to establish a non-renormalization theorem for the beta function β_g, stating that if the one-loop order contribution vanishes, then β_g will vanish to all orders of perturbation theory. As a by-product, the special case in which β_g is only of one-loop order, without further corrections, is also covered. The examples of the $N = 2, 4$ supersymmetric Yang–Mills theories are worked out in detail.

PACS numbers: 02.10.De, 03.70.+k, 11.10.-z, 11.15.-q, 11.30.Pb

1. Introduction

The search for ultraviolet finite renormalizable models has always been one of the most attractive and relevant aspect of quantum field theory. The requirement of a softer ultraviolet behaviour has motivated the construction of many models, including the Yang–Mills supersymmetric theories (SYM), the supergravities as well as the superstrings.

The ultraviolet finiteness is understood here as the vanishing, to all orders in the perturbative loop expansion, of the beta functions of the theory. This means that the dependence from the renormalization scale can be fully accounted by the unphysical anomalous dimensions of the field amplitudes which are, in general, nonvanishing.

So far, many ultraviolet finite theories have been given in different space-time dimensions. For instance, the Wess–Zumino–Witten models [1] and the $N = (4, 0)$ supersymmetric σ-model [2] are examples of two-dimensional theories which turn out to be conformal and superconformal, respectively.

In three space-time dimensions, the so-called topologically massive Yang–Mills theory, obtained by adding the Chern–Simons action to the Yang–Mills term, is one of the most celebrated examples of a fully[4] finite theory [3–5] with applications in QCD at nonzero temperature. Also, the pure Chern–Simons theory is known to have vanishing beta function and field anomalous dimensions to all orders of perturbation theory [6–10]. Its topological nature has allowed to use perturbative techniques to evaluate topological invariants of knots theory [11]. The beta function corresponding to the Chern–Simons coefficient also vanishes in the presence of matter [12–15]. More generally, in the Abelian case this coefficient is known to be strongly constrained by the Coleman–Hill theorem [16], implying that it can receive at most one-loop finite corrections. We remark that the one-loop induced Chern–Simons coefficient has an important physical meaning, indeed identifying the transverse conductivity. In addition, as shown in [17], this coefficient turns out to be quantized by a topological argument. It is worth mentioning here that, recently, the Coleman–Hill theorem has been extended to the non-Abelian case [18].

Turning now to four dimensions, the supersymmetric gauge theories certainly display a unique ultraviolet behaviour, leading in some cases to finite renormalizable field theories. This is the case of $N = 4$ SYM, which provided the first example of a four-dimensional superconformal gauge theory [19, 20]. Concerning the $N = 2$ SYM, although it is not ultraviolet finite, its beta function obeys a remarkable nonrenormalization theorem, stating that it receives at most one-loop contributions [21, 22]. In the case of the $N = 1$ SYM a set of necessary and sufficient conditions for the vanishing of the beta function to all orders of perturbation theory has been established [23], making it possible to classify the $N = 1$ finite SYM theories.

Examples of higher-dimensional finite field theories are provided by the BF models [24], which belong to the class of the Schwarz-type topological field theories [25].

The ultraviolet finiteness of the above-mentioned theories was checked first by explicit loop calculations [3–5, 7, 12, 19, 21], and later proven, to all orders of perturbation theory, by using a suitable set of Ward identities characterizing the symmetry content of each model.

For instance, in the case of the $(4, 0)$ two-dimensional σ model the use of the BRST technique has allowed for a regularization-independent proof of the absence of the superconformal anomaly [26]. A BRST approach has also been employed in the case of the Wess–Zumino–Witten models [27] and of the four-dimensional $N = 4$ SYM [28, 29].

Concerning the $N = 2$ SYM, the proof of the nonrenormalization theorem of its beta function given in [30] is based on a key relationship between the whole action of $N = 2$ and a local gauge invariant polynomial which turns out to have vanishing anomalous dimension. A different proof of this theorem is available also within the context of the harmonic superspace [31].

A detailed analysis of the quantum properties of the supercurrent multiplet is at the basis of the finiteness conditions for $N = 1$ SYM theories [23].

The vanishing of the beta function for the topological field theories can be proven in a rather general way by making use of an additional nonanomalous symmetry, called vector supersymmetry, present in both Schwarz- and Witten-type theories [32, 33]. The existence of this further symmetry relies on the fact that the energy–momentum tensor of the topological

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[4] In this case the anomalous dimensions of the fields vanish as well.

theories can be cast in the form of a pure BRST variation. We also underline that the trace of the energy–momentum tensor, whose integrated quantum extension is directly related to the beta function, can be characterized by a set of Ward identities based on a local formulation of the dilation invariance [8, 14, 15, 34, 35]. This approach has been successfully applied to pure Chern–Simons [8] and to topologically massive Yang–Mills [14, 15]. In this latter case a different proof of the finiteness has been given in [36], using a cohomological argument for a generalized class of Yang–Mills theories.

Besides the use of Ward identities, the reduction of couplings introduced by Oehme and Zimmermann [37] provides a very powerful and original method in order to reduce the number of independent coupling constants present in a given model. The requirement that the reduced theory has fewer independent couplings leads to a nontrivial set of reduction equations, relating the various beta functions. Although some of the relationships between the couplings can be associated to the existence of symmetries, one has to observe that the solutions of the reduction equations do not always seem to correspond to any known invariance [37].

The aim of this work is to present a purely algebraic criterion, of general applicability, for the ultraviolet finiteness. The approach relies on the BRST cohomology [38] and exploits the set of descent equations following from the Wess–Zumino consistency condition. It turns out indeed that, in some cases, these equations allow one to establish a one-to-one correspondence between the quantized action of a given model and a local field polynomial, belonging to the cohomology of the BRST operator in the lowest level of the descent equations. As a consequence, the beta function of the theory can be proven to be related to the anomalous dimension of this polynomial. The absence of this anomalous dimension therefore entails a nonrenormalization theorem for the beta function. This theorem states that if the beta function vanishes at one-loop order, it will vanish to all orders of perturbation theory, implying the ultraviolet finiteness of the model. As a by-product, it will be also possible to cover the case in which the beta function happens to be only of one-loop order, without any further corrections. The advantage of this approach is that the anomalous dimension of the field polynomial to which the quantized action is related, is easier to control than the proper beta function thanks to the existence of additional Ward identities as for instance the ghost equation [10], always present in the Yang–Mills type theories in the Landau gauge.

The paper is organized as follows. In section 2 the general assumptions needed for the finiteness theorem are discussed. Section 3 is devoted to the proof of the theorem, including the analysis of the absence of higher-order corrections for the beta function. In section 4 several examples will be worked out. These include the case of Chern–Simons theory coupled to matter, the $N = 2$ and the 4 SYM theories in four dimensions. Finally, in section 5 we summarize our main results, presenting the conclusion.

2. The general set up

2.1. Classical aspects

Let us start by fixing the notations and by specifying the classical and quantum assumptions about the structure of the models which will be considered throughout. We shall work in a flat D-dimensional Euclidean space-time equipped with a set of fields generically denoted by $\{\Phi^i\}$, i labelling the different kinds of fields needed to properly quantize the model, i.e. gauge fields, matter fields, ghosts, ghosts for ghosts, etc. According to the Batalin–Vilkovisky quantization procedure [39], for each field Φ^i with ghost number $\mathcal{N}_{\Phi^i}$ and dimension d_{Φ^i}, one introduces a corresponding antifield Φ^{i*} with ghost number $-(1 + \mathcal{N}_{\Phi^i})$ and dimension $(D - d_{\Phi^i})$.

We shall start thus with a classical fully quantized action $\Sigma(\Phi^i, \Phi^{i*})$ which will be considered to be massless and, for simplicity, to have a unique coupling constant g. The action $\Sigma(\Phi^i, \Phi^{i*})$ is power-counting renormalizable and obeys the classical Slavnov–Taylor identity

$$\mathcal{S}(\Sigma) = \int \mathrm{d}^D x \, \frac{\delta \Sigma}{\delta \Phi^i} \frac{\delta \Sigma}{\delta \Phi^{i*}} = 0 \qquad (2.1)$$

which leads to the nilpotent linearized operator B_Σ

$$B_\Sigma = \int \mathrm{d}^D x \left(\frac{\delta \Sigma}{\delta \Phi^i} \frac{\delta}{\delta \Phi^{i*}} + \frac{\delta \Sigma}{\delta \Phi^{i*}} \frac{\delta}{\delta \Phi^i} \right) \qquad B_\Sigma B_\Sigma = 0. \qquad (2.2)$$

Concerning the dependence from the coupling constant g, we shall make use of the following parametrization:

$$\Sigma = \frac{1}{g^2} \int \mathrm{d}^D x \, \mathcal{L}_{\mathrm{inv}} + \Sigma_{\mathrm{gf}} + \Sigma_{\Phi^*} \qquad (2.3)$$

where $\mathcal{L}_{\mathrm{inv}}$ is the classical invariant Lagrangian identified as the part of Σ which is independent from the antifields, the ghosts and the Lagrange multipliers, entering respectively the gauge-fixing term Σ_{gf} and the antifield action Σ_{Φ^*}. As is well known, with this parametrization an L-loop Feynman diagram behaves as $g^{2(L-1)}$.

Now differentiating the Slavnov–Taylor identity (2.1) with respect to the coupling constant g, we obtain the equation

$$B_\Sigma \frac{\partial \Sigma}{\partial g} = 0 \qquad (2.4)$$

showing that $\partial \Sigma / \partial g$ is an invariant cocycle. Actually, according to the requirement that g is a physical parameter of the theory, the cocycle $\partial \Sigma / \partial g$ turns out to be nontrivial[5], identifying therefore the cohomology of the operator B_Σ in the sector of the integrated local polynomials with ghost number zero and dimension D.

Owing to the parametrization (2.3), it follows that

$$\frac{\partial \Sigma}{\partial g} = -\frac{2}{g^3} \int \omega_D^0 + B_\Sigma \Delta^{-1} \qquad (2.5)$$

where

$$\omega_D^0 = \mathrm{d}^D x \, \mathcal{L}_{\mathrm{inv}} + (\Phi^* - \text{dependent terms}) \qquad (2.6)$$

is a nonintegrated field polynomial with form degree D and zero ghost number and Δ^{-1} is a trivial integrated cocycle with negative ghost number. The appearance of possible antifields-dependent terms in the right-hand side of equation (2.6) accounts for the case in which one has to deal with open gauge algebras, which close only up to equations of motion. As we shall see, this will be the case of $N = 2$ and 4 SYM.

Hence, the integrated consistency condition

$$B_\Sigma \int \omega_D^0 = 0 \qquad (2.7)$$

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[5] It can be proven [40] that physical quantities, such as the Green functions of gauge invariant operators, are independent from a parameter α for which $\partial \Sigma / \partial \alpha$ is trivial, i.e. $\partial_\alpha \Sigma = B_\Sigma \Xi$ for some local polynomial Ξ. Such a parameter is called a gauge parameter.

can be translated at the nonintegrated level, giving rise to the following set of descent equations [40]:

$$
\begin{aligned}
B_\Sigma\, \omega_D^0 + d\omega_{D-1}^1 &= 0 \\
B_\Sigma\, \omega_{D-1}^1 + d\omega_{D-2}^2 &= 0 \\
&\cdots \\
B_\Sigma\, \omega_1^{D-1} + d\omega_0^D &= 0 \\
B_\Sigma\, \omega_0^D &= 0
\end{aligned}
\tag{2.8}
$$

with ω_{D-p}^p ($p = 0, \ldots, D$) being local field polynomials with form degree $(D - p)$ and ghost number p.

In what follows we will be interested in the class of models fulfilling the two assumptions given below:

(i) The cohomology of B_Σ is empty in all sectors with form degree $1 \leqslant p \leqslant D$.
(ii) The sector with form degree zero is nonvanishing, with a unique nontrivial element ω_0^D.

2.2. Quantum aspects

Concerning the quantum aspects, the first requirement is the absence of anomalies in the quantum extension of the Slavnov–Taylor identity, i.e.

$$
\begin{aligned}
\Gamma &= \Sigma + \mathrm{O}\,(\hbar) \\
\mathcal{S}\,(\Gamma) &= 0
\end{aligned}
\tag{2.9}
$$

where Γ is the 1PI generating functional.

As usual, the dependence of Γ from the renormalization point μ is governed by the Callan–Symanzik equation, whose generic form reads

$$
\mathcal{C}\,\Gamma = 0 \qquad \mathcal{C} \equiv \mu\frac{\partial}{\partial\mu} + \hbar\beta_g\frac{\partial}{\partial g} - \hbar\,\gamma_{\Phi^i}\,N_{\Phi^i}
\tag{2.10}
$$

where β_g is the beta function, γ_{Φ^i} stand for the anomalous dimensions of the fields, and N_{Φ^i} is the counting operator

$$
N_{\Phi^i} = \int \mathrm{d}^D x\,\left(\Phi^i\frac{\delta}{\delta\Phi^i} - \Phi^{i*}\frac{\delta}{\delta\Phi^{i*}}\right).
\tag{2.11}
$$

Following the procedure outlined in [40] and making use of the absence of anomalies in the Slavnov–Taylor identity (2.9), the cocycles $\left\{\omega_{D-p}^p; 0 \leqslant p \leqslant D\right\}$ can be promoted to quantum insertions $[\omega_{D-p}^p \cdot \Gamma]$ fulfilling the quantum version of the descent equations (2.8), i.e.

$$
\begin{aligned}
B_\Gamma[\omega_{D-p}^p \cdot \Gamma] + d[\omega_{D-p-1}^{p+1} \cdot \Gamma] &= 0 \\
B_\Gamma[\omega_0^D \cdot \Gamma] &= 0.
\end{aligned}
\tag{2.12}
$$

As shown in [40], the insertions $[\omega_{D-p}^p \cdot \Gamma]$ possess the same anomalous dimension γ_ω and obey the following Callan–Symanzik equation:

$$
\mathcal{C}[\omega_{D-p}^p \cdot \Gamma] + \hbar\gamma_\omega[\omega_{D-p}^p \cdot \Gamma] = \hbar\,B_\Gamma\,[\Xi_{D-p}^{p-1} \cdot \Gamma]
\tag{2.13}
$$

for some cohomologically trivial local polynomial Ξ_{D-p}^{p-1}.

The last important assumption which we shall require is that the anomalous dimension γ_ω of the insertion $[\omega_0^D \cdot \Gamma]$ vanishes, i.e. $\gamma_\omega = 0$. Thus

$$
\mathcal{C}[\omega_0^D \cdot \Gamma] = \hbar\,B_\Gamma[\Xi_0^{D-1} \cdot \Gamma]
\tag{2.14}
$$

which, of course, implies that

$$C\left[\int \omega_D^0 \cdot \Gamma\right] = \hbar\, B_\Gamma\left[\int \Xi_D^{-1} \cdot \Gamma\right].\tag{2.15}$$

In summary, we are dealing with a theory for which there exists a one-to-one relationship between the solutions ω_D^0 and ω_0^D corresponding to the top and bottom levels of the classical descent equations (2.8). In addition, besides the absence of anomalies in the Slavnov–Taylor identity, the quantum insertion $[\omega_0^D \cdot \Gamma]$ is required to have vanishing anomalous dimension, as stated by equation (2.14). These features will strongly constrain the beta function β_g. The main idea underlying this construction is that of exploiting the *one to one* correspondence between $\partial\Sigma/\partial g$ and the cocycle ω_0^D, which is not renormalized. It turns out that the nonrenormalization properties of ω_0^D affect directly all cocycles entering the descent equations (2.8), including, in particular, $\partial\Sigma/\partial g$ and its anomalous dimension, which is nothing but the beta function β_g.

3. The algebraic criterion for the ultraviolet finiteness

3.1. The finiteness theorem

The aim of this section is to cast the previous considerations into a precise statement about the beta function. Let $\beta_g^{(n)}$ denote the contribution of order $\hbar^n$ to the beta function β_g. The theory is specified by a quantum vertex functional $\Gamma = \Sigma + \mathrm{O}(\hbar)$, which fulfils all the above-mentioned assumptions, namely, the classical requirements (i) and (ii), and the quantum properties encoded in equations (2.9) and (2.15).

The following theorem holds.

Theorem. *If the one-loop order contribution $\beta_g^{(1)}$ vanishes, i.e. $\beta_g^{(1)} = 0$, then β_g vanishes to all orders of perturbation theory.*

Proof. In order to prove the theorem, let us first show that the following identity is valid:

$$\frac{\partial\Gamma}{\partial g} = -\frac{2}{g^3}\tilde{a}\left[\int \omega_D^0 \cdot \Gamma\right] + B_\Gamma[\Delta^{-1} \cdot \Gamma]\tag{3.16}$$

where $[\Delta^{-1} \cdot \Gamma]$ is an integrated insertion with negative ghost number and $\tilde{a}$ is a formal power series in $\hbar$

$$\tilde{a} = \left(1 + \sum_{j=1}^{\infty} \hbar^j a_j\right).\tag{3.17}$$

Notice also that the coefficients a_j are dimensionless since the theory is considered to be massless.

Equation (3.16) is indeed easily established by induction in $\hbar$. At the zeroth order it is obviously verified due to equation (2.5). Let us suppose then that it holds at the order $\hbar^n$, i.e.

$$\frac{\partial\Gamma}{\partial g} = -\frac{2}{g^3}\left(1 + \sum_{j=1}^{n} \hbar^j a_j\right)\left[\int \omega_D^0 \cdot \Gamma\right] + B_\Gamma[\hat{\Delta}^{-1} \cdot \Gamma] + \hbar^{n+1}\Theta_{n+1} + \mathrm{O}(\hbar^{n+2})\tag{3.18}$$

where, from the quantum action principle [40], Θ_{n+1} is an integrated local polynomial with ghost number zero which obeys the condition

$$B_\Sigma\Theta_{n+1} = 0\tag{3.19}$$

following from

$$B_\Gamma \frac{\partial \Gamma}{\partial g} = 0 \qquad B_\Gamma B_\Gamma = 0. \tag{3.20}$$

Therefore, taking into account that the unique nontrivial cohomology class of B_Σ with the same quantum numbers of the action is $\int \omega_D^0$, we get

$$\Theta_{n+1} = a_{n+1} \int \omega_D^0 + B_\Sigma \hat{\Theta}_{n+1}^{-1} \tag{3.21}$$

which establishes the validity of equation (3.16) at the order $\hbar^{n+1}$, and hence to all orders by induction.

Now, coming back to the proof of the theorem, we act with the Callan–Symanzik operator $\mathcal{C}$ on the equation (3.16). Making use of equations (2.10) and (2.15), and recalling the exact commutation relation

$$\mathcal{C} B_\Gamma - B_\Gamma \mathcal{C} = 0 \tag{3.22}$$

we get the condition

$$\left[\mathcal{C}, \frac{\partial}{\partial g} \right] \Gamma = -\left(\mathcal{C} \left(\frac{2}{g^3} \tilde{a} \right) \right) \left[\int \omega_D^0 \cdot \Gamma \right] + \hbar B_\Gamma [\Omega^{-1} \cdot \Gamma] \tag{3.23}$$

for some irrelevant trivial insertion $[\Omega^{-1} \cdot \Gamma]$ with negative ghost number. Working out the commutator in the left-hand side and observing that the dimensionless coefficients a_j do not depend on μ, we obtain

$$\left(\left(\frac{\partial}{\partial g} \beta_g \right) \frac{2}{g^3} \tilde{a} + \beta_g \frac{\partial}{\partial g} \left(\frac{2}{g^3} \tilde{a} \right) \right) \left[\int \omega_D^0 \cdot \Gamma \right] = B_\Gamma [\hat{\Omega}^{-1} \cdot \Gamma] \tag{3.24}$$

which, due to the fact that the insertion $\left[\int \omega_D^0 \cdot \Gamma \right]$ cannot be written as a pure B_Γ-variation, finally implies the condition

$$\left(\frac{\partial}{\partial g} \beta_g \right) \frac{2}{g^3} \tilde{a} + \beta_g \frac{\partial}{\partial g} \left(\frac{2}{g^3} \tilde{a} \right) = 0. \tag{3.25}$$

This equation expresses the content of the theorem, stating indeed that if the one-loop contribution to the beta function vanishes, $\beta_g^{(1)} = 0$, then $\beta_g = 0$.

For a better understanding of the equation (3.25) let us expand β_g and $\tilde{a}$ in powers of $\hbar$, yielding

Order 1.

$$g \frac{\partial \beta_g^{(1)}}{\partial g} - 3\beta_g^{(1)} = 0 \implies \beta_g^{(1)} \sim g^3. \tag{3.26}$$

Order 2.

$$\left(g \frac{\partial \beta_g^{(2)}}{\partial g} - 3\beta_g^{(2)} \right) + \beta_g^{(1)} g \frac{\partial a_1}{\partial g} = 0. \tag{3.27}$$

Order n.

$$\left(g\frac{\partial\beta_g^{(n)}}{\partial g} - 3\beta_g^{(n)}\right) + \sum_{i=1}^{n-1}\left(\left(g\frac{\partial\beta_g^{(n-i)}}{\partial g} - 3\beta_g^{(n-i)}\right)a_i + \beta_g^{(n-i)}\,g\frac{\partial a_i}{\partial g}\right) = 0. \tag{3.28}$$

It becomes apparent thus that if $\beta_g^{(1)} = 0$ in the above equations, then $\beta_g^{(n)} = 0$ for all n.

Before discussing the applications of this result, let us underline that the present set up also provides a simple algebraic understanding of the case in which β_g receives contributions only up to one-loop order as, for instance, in the $N = 2$ SYM. This will be the aim of the next section. $\qquad\square$

3.2. Absence of higher-order corrections

It is known that the beta function β_g depends on the renormalization scheme, only the first-order coefficient being universal [41]. However, for some theories it happens that β_g receives contributions only up to one-loop order. This statement means really that there exists renormalization schemes in which all the higher loop corrections vanish. These schemes can be identified in an algebraic way by the following proposition.

Proposition. *For any renormalization scheme in which the following identity holds:*

$$\frac{\partial\Gamma}{\partial g} = -\frac{2}{g^3}\left[\int\omega_D^0\cdot\Gamma\right] + B_\Gamma[\Delta^{-1}\cdot\Gamma] \tag{3.29}$$

for some integrated insertion $[\Delta^{-1}\cdot\Gamma]$, *then* β_g *has at most one-loop contributions.*

Proof. The equation (3.29) is equivalent to (3.16) with the requirement that now $a_j = 0$ for any j. Repeating therefore the same steps as before, the equation (3.25) becomes

$$g\frac{\partial\beta_g}{\partial g} - 3\beta_g = 0 \tag{3.30}$$

which implies that β_g has only one-loop contributions, i.e. $\beta_g \sim g^3$. The identity (3.29) will turn out to be very useful in the analysis of $N = 2$ SYM. $\qquad\square$

4. Applications

In this section we shall present some applications of the finiteness criterion discussed previously. Let us begin with the case of the three-dimensional non-Abelian Chern–Simons theory coupled to spinor matter.

4.1. Chern–Simons coupled to matter

The classical invariant action of the model is given by

$$S_{\text{inv}} = \int \mathrm{d}^3x\,\left(\frac{1}{2g^2}\varepsilon^{\mu\nu\rho}\,\mathrm{Tr}\,\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) + \mathrm{i}\overline{\Psi}\,\gamma^\mu\,D_\mu\Psi\right). \tag{4.31}$$

The gauge field A_μ belongs to the adjoint representation of a general compact Lie group G:

$$A_\mu(x) = A_\mu^a(x)\,\tau_a \tag{4.32}$$

where the matrices τ_a are the generators of the group, chosen to be anti-Hermitean:

$$[\tau_a, \tau_b] = f_{abc}\,\tau_c \qquad \mathrm{Tr}\,\tau_a\tau_b = \delta_{ab}. \tag{4.33}$$

Table 1. Dimension, ghost number and nature of the fields.

	A_μ	c	$\overline{c}$	b	$\overline{\Psi}$	Ψ	A_μ^*	c^*	Ψ^*	$\overline{\Psi}^*$
Dim.	1	0	1	1	1	1	2	3	2	2
N.Ghost	0	1	−1	0	0	0	−1	−2	−1	−1
Nature	C	A	A	C	A	A	A	C	C	C

The matter fields belong to some finite representation of G, the corresponding generators being denoted by T_a. Hence, for the covariant derivative we have

$$D_\mu \Psi = (\partial_\mu + A_\mu^a T_a)\Psi. \tag{4.34}$$

Adopting the Landau condition, the gauge-fixing term reads

$$S_{gf} = s\, \mathrm{Tr} \int \mathrm{d}^3x \; \overline{c}\partial^\mu A_\mu = \mathrm{Tr} \int \mathrm{d}^3x \; \left(b\partial^\mu A_\mu + \overline{c}\partial^\mu D_\mu c\right) \tag{4.35}$$

where $c, \overline{c}$ and b denote respectively, the Faddeev–Popov ghost, the antighost and the Lagrangian multiplier, all of them in the same representation as A_μ. The BRST operator s acts on the fields as follows:

$$\begin{aligned}
sA_\mu &= -D_\mu c = -\left(\partial_\mu c + [A_\mu, c]\right) \\
sc &= c^2 \\
s\Psi &= c^a T_a \Psi \\
s\overline{\Psi} &= \overline{\Psi}\, T_a\, c^a \\
s\overline{c} &= b \\
sb &= 0.
\end{aligned} \tag{4.36}$$

Coupling now the nonlinear BRST transformations to the antifields $A_\mu^*, c^*, \overline{\Psi}^*, \Psi^*$

$$S_{ext} = \int \mathrm{d}^3x \left(\mathrm{Tr}\left(-A_\mu^* D^\mu c + c^* c^2\right) + \overline{\Psi}^* c^a T_a\, \Psi - \overline{\Psi}\, T_a\, c^a\, \Psi^* \right) \tag{4.37}$$

it turns out that the fully quantized classical action Σ

$$\Sigma = S_{inv} + S_{gf} + S_{ext} \tag{4.38}$$

obeys the Slavnov–Taylor identity

$$\mathcal{S}(\Sigma) = \int \mathrm{d}^3x \left(\mathrm{Tr}\left(\frac{\delta\Sigma}{\delta A_\mu^*}\frac{\delta\Sigma}{\delta A^\mu} + \frac{\delta\Sigma}{\delta c^*}\frac{\delta\Sigma}{\delta c} + b\frac{\delta\Sigma}{\delta\overline{c}} \right) + \frac{\delta\Sigma}{\delta\overline{\Psi}^*}\frac{\delta\Sigma}{\delta\Psi} - \frac{\delta\Sigma}{\delta\Psi^*}\frac{\delta\Sigma}{\delta\overline{\Psi}} \right) = 0. \tag{4.39}$$

Accordingly, the nilpotent linearized operator B_Σ is given by

$$\begin{aligned}
B_\Sigma = \int \mathrm{d}^3x \Bigg(&\mathrm{Tr}\left(\frac{\delta\Sigma}{\delta A_\mu^*}\frac{\delta}{\delta A^\mu} + \frac{\delta\Sigma}{\delta A^\mu}\frac{\delta}{\delta A_\mu^*} + \frac{\delta\Sigma}{\delta c^*}\frac{\delta}{\delta c} + \frac{\delta\Sigma}{\delta c}\frac{\delta}{\delta c^*} + b\frac{\delta}{\delta\overline{c}} \right) \\
&+ \frac{\delta\Sigma}{\delta\overline{\Psi}^*}\frac{\delta}{\delta\Psi} + \frac{\delta\Sigma}{\delta\Psi}\frac{\delta}{\delta\overline{\Psi}^*} - \frac{\delta\Sigma}{\delta\Psi^*}\frac{\delta}{\delta\overline{\Psi}} - \frac{\delta\Sigma}{\delta\overline{\Psi}}\frac{\delta}{\delta\Psi^*} \Bigg).
\end{aligned} \tag{4.40}$$

For further use, the quantum numbers of all fields and antifields are displayed in table 1.

Having quantized the theory, let us turn to the characterization of the cohomology of B_Σ in the sector of the invariant counterterms:

$$B_\Sigma \Delta^0 = 0 \tag{4.41}$$

where Δ^0 is an integrated local polynomial with dimension three and zero ghost number. Setting

$$\Delta^0 = \int d^3x \, \omega^0 \tag{4.42}$$

we obtain the following set of descent equations:

$$\begin{aligned}
B_\Sigma \omega^0 &= \partial^\mu \omega^1_\mu \\
B_\Sigma \omega^1_\mu &= \partial^\nu \omega^2_{[\mu\nu]} \\
B_\Sigma \omega^2_{[\mu\nu]} &= \partial^\rho \omega^3_{[\mu\nu\rho]} \\
B_\Sigma \omega^3_{[\mu\nu\rho]} &= 0.
\end{aligned} \tag{4.43}$$

The unique nontrivial solution for $\omega^3_{[\mu\nu\rho]}$ is given by

$$\omega^3_{[\mu\nu\rho]} = \varsigma \, \varepsilon_{\mu\nu\rho} \, \tfrac{1}{3} \mathrm{Tr} \, c^3 \tag{4.44}$$

where ς is a constant parameter. The higher cocycles ω^0, ω^1_μ and $\omega^2_{[\mu\nu]}$ are easily worked out and found to be

$$\begin{aligned}
\omega^2_{[\mu\nu]} &= -\varsigma \, \varepsilon_{\mu\nu\rho} \, \mathrm{Tr} \, c \partial^\rho c \\
\omega^1_\mu &= \varsigma \, \varepsilon_{\mu\nu\rho} \, \mathrm{Tr} \, A^\nu \partial^\rho c \\
\omega^0 &= -\varsigma \, \varepsilon^{\mu\nu\rho} \, \mathrm{Tr} \, (A_\mu \partial_\nu A_\rho + \tfrac{2}{3} A_\mu A_\nu A_\rho).
\end{aligned} \tag{4.45}$$

Concerning possible contributions coming from the spinor fields and the antifields, it turns out by explicit inspection that they give rise only to cohomologically trivial solutions, as can be straightforwardly checked with the Dirac term appearing in the complete action Σ, namely

$$i\overline{\Psi}\gamma^\mu D_\mu \Psi = B_\Sigma(\overline{\Psi}\Psi^*). \tag{4.46}$$

The solution given in equations (4.44) and (4.45) is thus the most general nontrivial solution of the descent equations (4.43). Of course, one always has the freedom of adding trivial terms.

Acting now with $\partial/\partial g$ on the Slavnov–Taylor identity one obtains

$$B_\Sigma \frac{\partial \Sigma}{\partial g} = 0 \tag{4.47}$$

with

$$\frac{\partial \Sigma}{\partial g} = -\frac{1}{g^3} \int d^3x \, \varepsilon^{\mu\nu\rho} \, \mathrm{Tr} \left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right). \tag{4.48}$$

It becomes apparent therefore that $\partial\Sigma/\partial g$ coincides with Δ^0 by taking $\varsigma = 1/g^3$. In particular, $\partial\Sigma/\partial g$ identifies the unique nontrivial class of the cohomology of B_Σ in the sector of counterterms. Moreover, there exists a one-to-one relationship between $\partial\Sigma/\partial g$ and the ghost polynomial $\mathrm{Tr} \, c^3$, implying that all classical assumptions of the finiteness criterion are fulfilled. Concerning now the quantum aspects, we point out that the Slavnov–Taylor identity can be established for the vertex functional Γ, due to the well known absence of the gauge anomaly in three dimensions [40].

According then to the general set up, the last requirement to be satisfied in order to apply the finiteness theorem is to prove that the gauge invariant field polynomial $\mathrm{Tr} \, c^3$ can be promoted to a quantum insertion $\left[\mathrm{Tr} \, c^3 \cdot \Gamma\right]$ with vanishing anomalous dimension. This is ensured by the so-called ghost equation Ward identity [10, 40]

$$\int d^3x \left(\frac{\delta}{\delta c} + \left[\overline{c}, \frac{\delta}{\delta b} \right] \right) \Sigma = \Delta^{\mathrm{cl}} \tag{4.49}$$

where Δ^{cl} is a classical breaking

$$\Delta^{\text{cl}} = \int \mathrm{d}^3 x \left([A^*_\mu, A^\mu] - [c^*, c] + \left(\overline{\Psi}^* T^a \Psi + \overline{\Psi} \, T^a \, \Psi^* \right) \tau_a \right). \tag{4.50}$$

As shown in detail in [10, 40] the Ward identity (4.49) allows one to control the dependence of the theory from the Faddeev–Popov ghost, implying, in particular, the vanishing of the anomalous dimension of $[\text{Tr} \, c^3 \cdot \Gamma]$ to all orders of perturbation theory.

Concerning the one-loop behaviour of the beta function, it is worth remembering here that the ultraviolet finiteness of Chern–Simons at one-loop order, with or without matter, is a well known result, being checked in many ways by several authors (see for instance [7]). Therefore, according to the finiteness theorem, β_g vanishes to all orders of perturbation theory. This example shows in a rather simple way that a great amount of information on the beta function β_g follows from the knowledge of the anomalous dimension of the gauge invariant insertion $[\text{Tr} \, c^3 \cdot \Gamma]$.

4.2. $N = 2$ super-Yang–Mills

The nonrenormalization theorem of the beta function of $N = 2$ SYM, stating that β_g receives only one-loop contributions, has long been known [21, 22]. Recently, a purely algebraic proof of this result, based on BRST Ward identities, has been given in [30]. It can be considered as a highly nontrivial realization of the algebraic finiteness criterion. In this section we shall review the main steps of the proof within the present context.

In order to study the quantum properties of $N = 2$ we shall make use of the twisting procedure which allows one to replace the spinor indices of supersymmetry $(\alpha, \dot{\alpha})$ with Lorentz vector indices. The physical content of the theory is left unchanged, since the twist is a linear change of variables, and the twisted version is perturbatively indistinguishable from the original one. However, the use of the twisted variables considerably simplifies the analysis of the finiteness properties, allowing one to identify a subset of supercharges which is actually relevant to control the ultraviolet behaviour.

Let us begin by sketching the twisting procedure for the $N = 2$ SYM in the Wess–Zumino (WZ) gauge [30, 42]. The global symmetry group of $N = 2$ in four-dimensional flat Euclidean space-time is $SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1)_{\mathcal{R}}$, where $SU(2)_L \times SU(2)_R$ is the rotation group and $SU(2)_I$ and $U(1)_{\mathcal{R}}$ are the symmetry groups corresponding to the internal $SU(2)$-transformations and to the $\mathcal{R}$-symmetry. The twisting procedure consists of replacing the rotation group by $SU(2)_L \times SU(2)'_R$, where $SU(2)'_R$ is the diagonal sum of $SU(2)_R$ and $SU(2)_I$, allowing one to identify the internal indices with the spinor indices. The fields of the $N = 2$ vector multiplet in the WZ gauge are given by $(A_\mu, \psi^i_\alpha, \overline{\psi}^i_{\dot{\alpha}}, \phi, \overline{\phi})$, where $\psi^i_\alpha, \overline{\psi}^i_{\dot{\alpha}}$ are Weyl spinors with $i = 1, 2$ being the internal index of the fundamental representation of $SU(2)_I$, and $\phi, \overline{\phi}$ are complex scalars. All fields belong to the adjoint representation of the gauge group. Under the twisted group, these fields decompose as [30, 42]

$$\begin{aligned}
A_\mu &\to A_\mu & (\phi, \overline{\phi}) &\to (\phi, \overline{\phi}) \\
\psi^i_\alpha &\to (\eta, \, \chi_{\mu\nu}) & \overline{\psi}^i_{\dot{\alpha}} &\to \psi_\mu.
\end{aligned} \tag{4.51}$$

Notice that $(\psi_\mu, \chi_{\mu\nu}, \eta)$ anticommute due to their spinor nature, and $\chi_{\mu\nu}$ is a self-dual tensor field. The action of $N = 2$ SYM in terms of the twisted variables is found to be [30, 42]

$$S^{N=2} = \frac{1}{g^2} \text{Tr} \int \mathrm{d}^4 x \left(\frac{1}{2} F^+_{\mu\nu} F^{+\mu\nu} + \frac{1}{2} \overline{\phi} \{ \psi^\mu, \psi_\mu \} - \chi^{\mu\nu} (D_\mu \psi_\nu - D_\nu \psi_\mu)^+ + \eta D_\mu \psi^\mu \right.$$

$$\left. - \frac{1}{2} \overline{\phi} D_\mu D^\mu \phi - \frac{1}{2} \phi \{ \chi^{\mu\nu}, \chi_{\mu\nu} \} - \frac{1}{8} [\phi, \eta] \eta - \frac{1}{32} [\phi, \overline{\phi}][\phi, \overline{\phi}] \right) \tag{4.52}$$

where g is the *unique* coupling constant and

$$F^+_{\mu\nu} = F_{\mu\nu} + \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} \qquad \tilde{F}_{\mu\nu}{}^+ = \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{+\rho\sigma} = F^+_{\mu\nu}$$
$$(D_\mu\psi_\nu - D_\nu\psi_\mu)^+ = (D_\mu\psi_\nu - D_\nu\psi_\mu) + \tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}(D^\rho\psi^\sigma - D^\sigma\psi^\rho). \tag{4.53}$$

Also, it is easily seen that assigning to $\left(A_\mu, \psi_\mu, \chi_{\mu\nu}, \eta, \phi, \overline{\phi}\right)$ the following $\mathcal{R}$-charges $(0, -1, 1, -1, 2, -2)$, the expression (4.52) has vanishing total $\mathcal{R}$-charge.

The action $S^{N=2}$ is invariant under gauge transformations with infinitesimal parameter ζ:

$$\delta^g_\zeta A_\mu = -D_\mu\zeta = -(\partial_\mu\zeta + [A_\mu, \zeta])$$
$$\delta^g_\zeta \gamma = [\zeta, \gamma] \qquad \text{with} \quad \gamma = (\psi_\mu, \chi_{\mu\nu}, \eta, \phi, \overline{\phi}) \tag{4.54}$$

which lead to the usual BRST transformations, with $\delta^g_\zeta \to s$ and $\zeta \to c$, where c is the Faddeev–Popov ghost transforming as $sc = c^2$.

Concerning the supersymmetry generators $(\delta^\alpha_i, \overline{\delta}^i_{\dot\alpha})$ of the $N = 2$ superalgebra, it turns out that the twisting procedure gives rise to the following twisted generators: a scalar δ, a vector δ_μ and a self-dual tensor $\delta_{\mu\nu}$, which of course leave the action invariant. It is worth emphasizing that $S^{N=2}$ is uniquely fixed by the scalar δ and the vector δ_μ twisted charges. Due to this property, the tensor generator $\delta_{\mu\nu}$ will not be taken into further account, although its inclusion can be done straightforwardly.

In order to properly quantize the theory we collect all the generators (s, δ, δ_μ) into an extended operator Q, which turns out to be nilpotent on-shell and modulo the space-time translations

$$Q = s + \omega\delta + \varepsilon^\mu\delta_\mu \tag{4.55}$$
$$Q^2 = 0 + \omega\varepsilon^\mu\partial_\mu + \text{equations of motion} \tag{4.56}$$

where ω and ε^μ are global ghosts. The operator Q acts on the fields as

$$QA_\mu = -D_\mu c + \omega\psi_\mu + \frac{\varepsilon^\nu}{2}\chi_{\nu\mu} + \frac{\varepsilon_\mu}{8}\eta$$
$$Q\psi_\mu = \{c, \psi_\mu\} - \omega D_\mu\phi + \varepsilon^\nu\left(F_{\nu\mu} - \frac{1}{2}F^+_{\nu\mu}\right) - \frac{\varepsilon_\mu}{16}[\phi, \overline{\phi}]$$
$$Q\chi_{\sigma\tau} = \{c, \chi_{\sigma\tau}\} + \omega F^+_{\sigma\tau} + \frac{\varepsilon^\mu}{8}\left(\epsilon_{\mu\sigma\tau\nu} + g_{\mu\sigma}g_{\nu\tau} - g_{\mu\tau}g_{\nu\sigma}\right)D^\nu\overline{\phi}$$
$$Q\eta = \{c, \eta\} + \frac{\omega}{2}[\phi, \overline{\phi}] + \frac{\varepsilon^\mu}{2}D_\mu\overline{\phi}, \tag{4.57}$$
$$Q\phi = [c, \phi] - \varepsilon^\mu\psi_\mu$$
$$Q\overline{\phi} = [c, \overline{\phi}] + 2\omega\eta$$
$$Qc = c^2 - \omega^2\phi - \omega\varepsilon^\mu A_\mu + \frac{\varepsilon^2}{16}\overline{\phi}.$$

Following the Batalin–Vilkovisky procedure, for the complete gauge-fixed action we obtain [30, 42]

$$\Sigma = S^{N=2} + S_{\text{gf}} + S_{\text{ext}} \tag{4.58}$$

where S_{gf} is the gauge-fixing term in the Landau gauge and S_{ext} contains the coupling of the nonlinear transformations $Q\Phi_i$ to antifields $\Phi^*_i = (A^*_\mu, \psi^*_\mu, \tfrac{1}{2}\chi^*_{\mu\nu}, \eta^*, \phi^*, \overline{\phi}^*, c^*)$. They are

given by[6]

$$S_{\text{gf}} = Q \int \mathrm{d}^4x \, \mathrm{Tr} \, (\bar{c} \partial A)$$

$$S_{\text{ext}} = \mathrm{Tr} \int \mathrm{d}^4x \left(\Phi_i^* Q \Phi_i + \frac{g^2}{32} \left(4\omega^2 \chi^{*2} - 8\omega \varepsilon_\mu \chi^{*\mu\nu} \psi_\nu^* + \varepsilon^2 \psi^{*2} - (\varepsilon \psi^*)^2 \right) \right) \tag{4.59}$$

with

$$Q\bar{c} = b \qquad Qb = \omega \varepsilon^\mu \partial_\mu \bar{c} \tag{4.60}$$

where, as usual, $\bar{c}$, b denote the antighost and the Lagrange multiplier.

The complete action Σ thus satisfies the following Slavnov–Taylor identity:

$$\mathcal{S}(\Sigma) = \omega \varepsilon^\mu \Delta_\mu^{\text{cl}} \tag{4.61}$$

where

$$\mathcal{S}(\Sigma) = \mathrm{Tr} \int \mathrm{d}^4x \left(\frac{\delta \Sigma}{\delta \Phi_i^*} \frac{\delta \Sigma}{\delta \Phi_i} + b \frac{\delta \Sigma}{\delta \bar{c}} + \omega \varepsilon^\mu \partial_\mu \bar{c} \frac{\delta \Sigma}{\delta b} \right) \tag{4.62}$$

and Δ_μ^{cl} is an integrated local polynomial

$$\Delta_\mu^{\text{cl}} = \mathrm{Tr} \int \mathrm{d}^4x \left(c^* \partial_\mu c - \phi^* \partial_\mu \phi - A^{*\nu} \partial_\mu A_\nu - \psi^{*\nu} \partial_\mu \psi_\nu - \bar{\phi}^* \partial_\mu \bar{\phi} + \eta^* \partial_\mu \eta + \tfrac{1}{2} \chi^{*\nu\rho} \partial_\mu \chi_{\nu\rho} \right). \tag{4.63}$$

Notice that Δ_μ^{cl}, being linear in the quantum fields, is a classical breaking and will not be affected by the quantum corrections. From the Slavnov–Taylor identity it follows that the linearized operator B_Σ defined as

$$B_\Sigma = \mathrm{Tr} \int \mathrm{d}^4x \left(\frac{\delta \Sigma}{\delta \Phi_i^*} \frac{\delta}{\delta \Phi_i} + \frac{\delta \Sigma}{\delta \Phi_i} \frac{\delta}{\delta \Phi_i^*} + b \frac{\delta}{\delta \bar{c}} + \omega \varepsilon^\mu \partial_\mu \bar{c} \frac{\delta}{\delta b} \right) \tag{4.64}$$

turns out to be nilpotent modulo a total space-time derivative, namely

$$B_\Sigma B_\Sigma = \omega \varepsilon^\mu \partial_\mu. \tag{4.65}$$

The appearance of the space-time translation operator ∂_μ in the right-hand of equation (4.65) is due to the supersymmetric structure of the theory. Of course, the operator B_Σ can be considered nilpotent when acting on the space of the integrated local polynomials. Moreover, as we shall see in detail, the presence of the space-time derivative ∂_μ will give rise to a set of nonstandard descent equations which will turn out to constrain very strongly the possible nontrivial invariant counterterms. We will also be able to prove that these equations can be solved in a systematic way by using the twisted $N = 2$ supersymmetric algebra.

Proceeding as in the previous example, we act with the operator $\partial / \partial g$ on both sides of the Slavnov–Taylor identity (4.61). Observing then that the linear breaking term Δ_μ^{cl} does not depend on the coupling constant g, we get the condition

$$B_\Sigma \frac{\partial \Sigma}{\partial g} = 0 \tag{4.66}$$

which shows that $\partial \Sigma / \partial g$ is invariant under the action of B_Σ. It remains to prove that $\partial \Sigma / \partial g$ is nontrivial. We are led then to solve the consistency condition for the integrated invariant counterterms

$$B_\Sigma \int \mathrm{d}^4x \, \Omega^0 = 0 \tag{4.67}$$

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[6] The presence of terms quadratic in the antifields in S_{ext} is due to the fact that the operator Q is nilpotent up to the equations of motion.

where Ω^0 has the same quantum numbers as the classical action of $N = 2$. Due to equation (4.65), the integrated consistency condition (4.67) can be translated at the local level as

$$B_\Sigma \Omega^0 = \partial^\mu \Omega^1_\mu \tag{4.68}$$

where Ω^1_μ is a local polynomial with ghost number 1 and dimension 3. Applying now the operator B_Σ to both sides of (4.68) and making use of equation (4.65), one obtains the condition

$$\partial^\mu \left(B_\Sigma \Omega^1_\mu - \omega \varepsilon_\mu \Omega^0 \right) = 0 \tag{4.69}$$

which, due to the algebraic Poincaré lemma [40], implies

$$B_\Sigma \Omega^1_\mu = \omega \varepsilon_\mu \Omega^0 + \partial^\nu \Omega^2_{[\nu\mu]} \tag{4.70}$$

for some local polynomial $\Omega^2_{[\nu\mu]}$ antisymmetric in the Lorentz indices μ, ν and with ghost number 2. The procedure can be easily iterated, yielding the following set of descent equations:

$$\begin{aligned}
B_\Sigma \Omega^0 &= \partial^\mu \Omega^1_\mu \\
B_\Sigma \Omega^1_\mu &= \partial^\nu \Omega^2_{[\nu\mu]} + \omega \varepsilon_\mu \Omega^0 \\
B_\Sigma \Omega^2_{[\mu\nu]} &= \partial^\rho \Omega^3_{[\rho\mu\nu]} + \omega \varepsilon_\mu \Omega^1_\nu - \omega \varepsilon_\nu \Omega^1_\mu \\
B_\Sigma \Omega^3_{[\mu\nu\rho]} &= \partial^\sigma \Omega^4_{[\sigma\mu\nu\rho]} + \omega \varepsilon_\mu \Omega^2_{[\nu\rho]} + \omega \varepsilon_\rho \Omega^2_{[\mu\nu]} + \omega \varepsilon_\nu \Omega^2_{[\rho\mu]} \\
B_\Sigma \Omega^4_{[\mu\nu\rho\sigma]} &= \omega \varepsilon_\mu \Omega^3_{[\nu\rho\sigma]} - \omega \varepsilon_\sigma \Omega^3_{[\mu\nu\rho]} + \omega \varepsilon_\rho \Omega^3_{[\sigma\mu\nu]} - \omega \varepsilon_\nu \Omega^3_{[\rho\sigma\mu]}.
\end{aligned} \tag{4.71}$$

We observe that these equations are of a nonstandard type, as the cocycles with lower ghost number appear in the equations of those with higher ghost number, turning the system (4.71) nontrivial. We remark that the last equation for $\Omega^4_{[\mu\nu\rho\sigma]}$ is not homogeneous, a property which strongly constrains the possible solutions. Actually, it is possible to solve the system (4.71) in a rather direct way by making use of the $N = 2$ structure. To this end we introduce the operator

$$\mathcal{W}_\mu = \frac{1}{\omega} \left[\frac{\partial}{\partial \varepsilon^\mu}, B_\Sigma \right] \tag{4.72}$$

which obeys the relations

$$\{\mathcal{W}_\mu, B_\Sigma\} = \partial_\mu \qquad \{\mathcal{W}_\mu, \mathcal{W}_\nu\} = 0. \tag{4.73}$$

This algebra is typical of topological quantum field theories [32, 33]. In particular, as shown in [43], that the decomposition (4.73) allows one to make use of $\mathcal{W}_\mu$ as a climbing-up operator for the descent equations (4.71). It turns out, in fact, that the nontrivial solution of the system is

$$\Omega^0 = \frac{1}{4!} \mathcal{W}^\mu \mathcal{W}^\nu \mathcal{W}^\rho \mathcal{W}^\sigma \Omega^4_{[\sigma\rho\nu\mu]} \qquad \Omega^1_\mu = \frac{1}{3!} \mathcal{W}^\nu \mathcal{W}^\rho \mathcal{W}^\sigma \Omega^4_{[\sigma\rho\nu\mu]}$$

$$\Omega^2_{[\mu\nu]} = \frac{1}{2!} \mathcal{W}^\rho \mathcal{W}^\sigma \Omega^4_{[\sigma\rho\mu\nu]} \qquad \Omega^3_{[\mu\nu\rho]} = \mathcal{W}^\sigma \Omega^4_{[\sigma\mu\nu\rho]} \tag{4.74}$$

with $\Omega^4_{[\mu\nu\rho\sigma]}$ given by

$$\Omega^4_{[\mu\nu\rho\sigma]} = \omega^4 \varepsilon_{\mu\nu\rho\sigma} \operatorname{Tr} \phi^2. \tag{4.75}$$

From equations (4.74) the usefulness of the operator $\mathcal{W}_\mu$ now becomes apparent. Recalling thus that the cocycle Ω^0 has the same quantum numbers of the $N = 2$ Lagrangian, the following relation holds:

$$\frac{\partial \Sigma}{\partial g} = \frac{2\omega^4}{3g^3} \varepsilon^{\mu\nu\rho\sigma} \mathcal{W}_\mu \mathcal{W}_\nu \mathcal{W}_\rho \mathcal{W}_\sigma \int d^4x \operatorname{Tr} \frac{\phi^2}{2} + B_\Sigma \Xi^{-1} \tag{4.76}$$

for some irrelevant trivial Ξ^{-1}. This equation shows that there is a one-to-one relationship between the solution of the lowest level of the descent equations (4.71) and the action of $N = 2$,

so that the classical assumptions (i) and (ii) of section 2 are satisfied. Equation (4.76) implies that the ultraviolet behaviour of $N = 2$ can be traced back to gauge invariant polynomial $\text{Tr}\,\phi^2$, which plays the role of a kind of perturbative prepotential.

Concerning the quantum aspects, the Slavnov–Taylor identity (4.61) can be extended to the quantum level without anomalies [44]. Also, the construction given in [40] can be generalized to the set of descent equations (4.71), with the result that the cocycles Ω^0, Ω^1_μ, $\Omega^2_{[\mu\nu]}$, $\Omega^3_{[\mu\nu\rho]}$, $\Omega^4_{[\mu\nu\rho\sigma]}$ can be promoted to quantum insertions with the same anomalous dimension. Finally, the last requirement in order to apply the finiteness criterion is to establish the vanishing of the anomalous dimension of the insertion $[\text{Tr}\,\phi^2 \cdot \Gamma]$. This important property has indeed been proven in [30]. Without entering into further details, we limit ourselves here to remark that the proof of the vanishing of the anomalous dimension of $[\text{Tr}\,\phi^2 \cdot \Gamma]$ stems from a Ward identity relating $\text{Tr}\,\phi^2$ to the gauge invariant polynomial $\text{Tr}(-3\omega^2 c\phi + c^3)/\omega^4$, whose anomalous dimension vanishes due to the ghost equation [30]. In turn, this implies that $[\text{Tr}\,\phi^2 \cdot \Gamma]$ has vanishing anomalous dimension as well. Moreover, in the present case, it has been possible to prove that the classical equation (4.76) can be extended as it stands at the quantum level [30], yielding the remarkable equation

$$\frac{\partial \Gamma}{\partial g} = \frac{2\omega^4}{3g^3} \int \mathrm{d}^4 x \left[\left(\mathcal{W}^4 \, \text{Tr}\, \frac{\phi^2}{2} \right) \cdot \Gamma \right] + B_\Gamma [\Xi^{-1} \cdot \Gamma] \tag{4.77}$$

with $\mathcal{W}^4 = \varepsilon^{\mu\nu\rho\sigma} \mathcal{W}_\mu \mathcal{W}_\nu \mathcal{W}_\rho \mathcal{W}_\sigma$.

We observe that this equation has the form of (3.29), implying, in particular, the absence of the coefficients $\tilde{a}$ of equation (3.17). Therefore, the proposition of section 3.2 applies with the result that the beta function of $N = 2$ SYM is indeed of one-loop order only, i.e. $\beta_g \sim g^3$.

4.3. $N = 4$ super Yang–Mills

The case of the $N = 4$ SYM can be treated in a way similar to $N = 2$. Let us begin by describing how the twisting procedure can be applied. The global symmetry group of $N = 4$ SYM theory in Euclidean space-time is $SU(2)_L \times SU(2)_R \times SU(4)$, where $SU(2)_L \times SU(2)_R$ is the rotation group and $SU(4)$ the internal symmetry group of $N = 4$. Hence the twist operation can be performed in more than one way, depending on how the internal symmetry group is combined with the rotation group [45]. We shall follow the procedure of Vafa and Witten [46], in which the $SU(4)$ is split as $SU(2)_F \times SU(2)_I$, so that the twisted global symmetry group turns out to be $SU(2)'_L \times SU(2)_R \times SU(2)_F$, where $SU(2)'_L = \text{diag}\,(SU(2)_L \oplus SU(2)_I)$ and $SU(2)_F$ is a residual internal symmetry group. The fields of the $N = 4$ multiplet are given by $(A_\mu, \lambda^\alpha_u, \overline{\lambda}^u_{\dot\alpha}, \Phi_{uv})$, where $(u, v = 1, \dots, 4)$ are indices of the fundamental representation of $SU(4)$, and the six real scalar fields of the model are collected into the antisymmetric and self-conjugate tensor Φ_{uv}. Under the twisted group, these fields decompose as

$$\begin{aligned} A_\mu &\to A_\mu & \overline{\lambda}^u_{\dot\alpha} &\to \psi^i_\mu \\ \lambda^\alpha_u &\to \eta^i,\ \chi^i_{\mu\nu} & \Phi_{uv} &\to B_{\mu\nu},\ \phi^{ij} \end{aligned} \tag{4.78}$$

where $(i, j = 1, 2)$ are indices of the residual isospin group $SU(2)_F$, ϕ^{ij} is a symmetric tensor, and $\chi^i_{\mu\nu}$, $B_{\mu\nu}$ are self-dual with respect to the Lorentz indices. Since in our analysis manifest isospin invariance is not needed, we further explicit the $SU(2)_F$ doublets as $\psi^i_\mu = (\psi_\mu, \chi_\mu)$, $\eta^i = (\eta, \xi)$, $\chi^i_{\mu\nu} = (\chi_{\mu\nu}, \psi_{\mu\nu})$ and the triplet as $\phi^{ij} = (\phi, \overline{\phi}, \tau)$. The action of $N = 4$ in

terms of the twisted fields is given by[7]

$$S^{N=4} = \frac{1}{g^2} \operatorname{Tr} \int d^4x \left(D_\mu \phi D^\mu \overline{\phi} + i\psi_\mu D_\nu \chi^{\mu\nu} + i\chi_\mu D_\nu \psi^{\mu\nu} - \chi_\mu D^\mu \xi \right.$$

$$+ \psi_\mu D^\mu \eta - i\overline{\phi}\{\psi_{\mu\nu}, \psi^{\mu\nu}\} + i\phi\{\chi_{\mu\nu}, \chi^{\mu\nu}\} + i\tau\{\psi_{\mu\nu}, \chi^{\mu\nu}\}$$

$$- \{\psi_{\mu\nu}, \chi^{\mu\rho}\}B^\nu_\rho - i\chi_{\mu\nu}[\xi, B^{\mu\nu}] - i\psi_{\mu\nu}[\eta, B^{\mu\nu}] + 4i\overline{\phi}\{\xi, \xi\}$$

$$- 4i\,\phi\,\{\eta, \eta\} + 4i\,\tau\,\{\xi, \eta\} + \psi_\mu\,[\chi_\nu, B^{\mu\nu}] + i\,\phi\,\{\chi_\mu, \chi^\mu\}$$

$$- i\overline{\phi}\,\{\psi_\mu, \psi^\mu\} - i\psi_\mu\,[\chi^\mu, \tau] - 4\,[\phi, \overline{\phi}]\,[\phi, \overline{\phi}] + 4[\phi, \tau][\overline{\phi}, \tau]$$

$$+ [\phi, B_{\mu\nu}]\,[\overline{\phi}, B^{\mu\nu}] - H^\mu(H_\mu - D_\mu\tau + i\,D^\nu B_{\mu\nu})$$

$$\left. + H^{\mu\nu}\left(-H_{\mu\nu} + \frac{i}{4}F^+_{\mu\nu} - \frac{1}{2}[B_{\mu\rho}, B^\rho_{\ \nu}] - i[B_{\mu\nu}, \tau] \right) \right) \tag{4.79}$$

where g is the unique coupling constant and $H_{\mu\nu}$, H_μ are auxiliary fields, with $H_{\mu\nu}$ self-dual.

Concerning the generators $(\delta^\alpha_u, \overline{\delta}^u_{\dot{\alpha}})$ of the $N = 4$ superalgebra, it turns out that the twisting procedure gives rise to the following twisted charges [47]: two scalars, δ^+ and δ^-, two vectors, δ^+_μ and δ^-_μ, and two self-dual tensors $\delta^+_{\mu\nu}$ and $\delta^-_{\mu\nu}$. Of course, all twisted generators leave the action (4.79) invariant. It is worth emphasizing that, as proven in [29], the action $S^{N=4}$ is uniquely fixed by the two vector generators δ^+_μ, δ^-_μ and by the scalar charge δ^+. In other words, the requirement of invariance under δ^+_μ, δ^-_μ and δ^+ fixes all the relative numerical coefficients of the various terms of the action (4.79). Thus, as done in the case of $N = 2$, the tensorial transformations $\delta^+_{\mu\nu}$, $\delta^-_{\mu\nu}$ will not be taken into account. The action of the twisted δ^+ generator on the fields reads

$$\begin{aligned}
&\delta^+ A_\mu = \psi_\mu && \delta^+ \tau = \xi && \delta^+ \psi_\mu = D_\mu\phi \\
&\delta^+ \phi = 0 && \delta^+ \chi_\mu = H_\mu && \delta^+ \xi = i\,[\tau, \phi] \\
&\delta^+ \overline{\phi} = -\eta && \delta^+ B_{\mu\nu} = \psi_{\mu\nu} && \delta^+ \eta = i[\phi, \overline{\phi}] \\
&\delta^+ \chi_{\mu\nu} = H_{\mu\nu} && \delta^+ \psi_{\mu\nu} = i[B_{\mu\nu}, \phi] \\
&\delta^+ H_\mu = i[\chi_\mu, \phi] && \delta^+ H_{\mu\nu} = i[\chi_{\mu\nu}, \phi].
\end{aligned} \tag{4.80}$$

In the first column of equation (4.80) we recognize the scalar transformations of the twisted $N = 2$ subalgebra in the presence of the auxiliary field $H_{\mu\nu}$. For δ^- one gets

$$\begin{aligned}
&\delta^- A_\mu = \chi_\mu && \delta^- \tau = -\eta && \delta^- \chi_\mu = -D_\mu\overline{\phi} \\
&\delta^- \psi_\mu = -H_\mu + D_\mu\tau && \delta^- \overline{\phi} = 0 && \delta^- \eta = i[\tau, \overline{\phi}] \\
&\delta^- \phi = -\xi && \delta^- \chi_{\mu\nu} = i[B_{\mu\nu}, \overline{\phi}] && \delta^- \xi = i[\phi, \overline{\phi}] \\
&\delta^- B_{\mu\nu} = -\chi_{\mu\nu} && \delta^- \psi_{\mu\nu} = H_{\mu\nu} + i[B_{\mu\nu}, \tau] \\
&\delta^- H_{\mu\nu} = -i[\psi_{\mu\nu}, \overline{\phi}] + i[\chi_{\mu\nu}, \tau] + i[B_{\mu\nu}, \eta] \\
&\delta^- H_\mu = -D_\mu\eta + i[\psi_\mu, \overline{\phi}] + i[\chi_\mu, \tau].
\end{aligned} \tag{4.81}$$

Analogously, for the vector transformations δ^+_μ and δ^-_μ one obtains

$$\begin{aligned}
&\delta^+_\mu A_\nu = -4i\chi_{\mu\nu} - 4g_{\mu\nu}\eta && \delta^+_\mu \tau = \chi_\mu && \delta^+_\mu \phi = \psi_\mu \\
&\delta^+_\mu \overline{\phi} = 0 && \delta^+_\mu \xi = D_\mu\tau - H_\mu && \delta^+_\mu \eta = -D_\mu\overline{\phi} \\
&\delta^+_\mu B_{\nu\rho} = -i\theta_{\mu\nu\rho\lambda}\chi^\lambda && \delta^+_\mu \psi_{\nu\rho} = D_\mu B_{\nu\rho} + i\theta_{\mu\nu\rho\lambda}H^\lambda \\
&\delta^+_\mu \chi_{\nu\rho} = i\theta_{\mu\nu\rho\lambda}D^\lambda\overline{\phi} && \delta^+_\mu \chi_\nu = -4[B_{\mu\nu}, \overline{\phi}] + 4ig_{\mu\nu}[\tau, \overline{\phi}] \\
&\delta^+_\mu \psi_\nu = 4iH_{\mu\nu} + F_{\mu\nu} - 4ig_{\mu\nu}[\overline{\phi}, \phi] \\
&\delta^+_\mu H_{\nu\rho} = D_\mu\chi_{\nu\rho} + \theta_{\mu\nu\rho\lambda}[\psi^\lambda, \overline{\phi}] + i\theta_{\mu\nu\rho\lambda}D^\lambda\eta \\
&\delta^+_\mu H_\nu = D_\mu\chi_\nu + 4[\eta, B_{\mu\nu}] + 4[\psi_{\mu\nu}, \overline{\phi}] - 4ig_{\mu\nu}[\eta, \tau] - 4ig_{\mu\nu}[\xi, \overline{\phi}]
\end{aligned} \tag{4.82}$$

---

[7] The group generators are chosen here to be Hermitian.

and

$$\delta^-_\mu A_\nu = -4\mathrm{i}\psi_{\mu\nu} + 4g_{\mu\nu}\xi \qquad \delta^-_\mu \tau = \psi_\mu \qquad \delta^-_\mu \phi = 0$$

$$\delta^-_\mu \bar\phi = -\chi_\mu \qquad \delta^-_\mu \xi = -D_\mu\phi \qquad \delta^-_\mu \eta = -H_\mu$$

$$\delta^-_\mu B_{\nu\rho} = +\mathrm{i}\theta_{\mu\nu\rho\lambda}\psi^\lambda \qquad \delta^-_\mu \psi_\nu = -4[B_{\mu\nu}, \phi] - 4\mathrm{i}g_{\mu\nu}[\tau, \phi] \qquad (4.83)$$

$$\delta^-_\mu \psi_{\nu\rho} = -\mathrm{i}\theta_{\mu\nu\rho\lambda}D^\lambda\phi \qquad \delta^-_\mu \chi_{\nu\rho} = -D_\mu B_{\nu\rho} - \mathrm{i}\theta_{\mu\nu\rho\lambda}H^\lambda + \mathrm{i}\theta_{\mu\nu\rho\lambda}D^\lambda\tau$$

$$\delta^-_\mu \chi_\nu = 4\mathrm{i}H_{\mu\nu} + F_{\mu\nu} + 4\mathrm{i}g_{\mu\nu}[\bar\phi, \phi] - 4[B_{\mu\nu}, \tau]$$

$$\delta^-_\mu H_{\nu\rho} = D_\mu\psi_{\nu\rho} + \theta_{\mu\nu\rho\lambda}([\psi^\lambda, \tau] - [\chi^\lambda, \phi] - \mathrm{i}D^\lambda\xi) + \mathrm{i}[\psi_\mu, B_{\nu\rho}]$$

$$\delta^-_\mu H_\nu = -D_\mu\psi_\nu + D_\nu\psi_\mu + 4[\psi_{\mu\nu}, \tau] - 4[\xi, B_{\mu\nu}] + 4[\chi_{\mu\nu}, \phi] + 4\mathrm{i}g_{\mu\nu}[\eta, \phi]$$

where $\theta_{\mu\nu\rho\sigma}$ denotes the combination

$$\theta_{\mu\nu\rho\sigma} = \varepsilon_{\mu\nu\rho\sigma} + g_{\mu\nu}g_{\rho\sigma} - g_{\mu\rho}g_{\nu\sigma} = 4\Pi^+_{\mu\sigma\nu\rho} \qquad (4.84)$$

where $\Pi^+_{\mu\sigma\nu\rho}$ is the projector on self-dual two-forms. Let us also give here the algebraic relations among the twisted generators, i.e.

$$\{\delta^+, \delta^+\} = \delta^g_{-2\phi} \qquad \{\delta^+_\mu, \delta^+\} = \partial_\mu + \delta^g_{A_\mu}$$

$$\{\delta^-, \delta^-\} = \delta^g_{2\bar\phi} \qquad \{\delta^-_\mu, \delta^-\} = \partial_\mu + \delta^g_{A_\mu}$$

$$\{\delta^+, \delta^-\} = \delta^g_{-\tau} \qquad \{\delta^+_\mu, \delta^-\} = 0 \qquad (4.85)$$

$$\{\delta^-_\mu, \delta^+\} = 0 \qquad \{\delta^+_\mu, \delta^+_\nu\} = \delta^g_{-8g_{\mu\nu}\bar\phi}$$

$$\{\delta^-_\mu, \delta^-_\nu\} = \delta^g_{8g_{\mu\nu}\phi} \qquad \{\delta^+_\mu, \delta^-_\nu\} = \delta^g_{-4\mathrm{i}B_{\mu\nu}-4g_{\mu\nu}\tau} + \text{equations of motion}$$

where δ^g_γ denotes a gauge transformation with parameter γ.

In order to quantize the theory, we proceed as before and introduce a generalized BRST operator Q which collects all the symmetry generators

$$Q = s + \omega^+\delta^+ + \omega^-\delta^- + \varepsilon^{+\mu}\delta^+_\mu + \varepsilon^{-\mu}\delta^-_\mu \qquad (4.86)$$

where s is the ordinary BRST operator for the gauge transformations, and ω^+, ω^-, $\varepsilon^{+\mu}, \varepsilon^{-\mu}$ are global ghosts [29]. Defining the action of Q on the Faddeev–Popov ghost c as

$$Qc = \mathrm{i}c^2 + (\omega^{+2} - 4\varepsilon^{-2})\phi + (4\varepsilon^{+2} - \omega^{-2})\bar\phi + (\omega^+\omega^- + 4\varepsilon^{+\mu}\varepsilon^-_\mu)\tau$$

$$+4\mathrm{i}\varepsilon^{+\mu}\varepsilon^{-\nu}B_{\mu\nu} - (\omega^+\varepsilon^{+\mu} + \omega^-\varepsilon^{-\mu})A_\mu \qquad (4.87)$$

it follows that the operator Q turns out to be nilpotent on shell and modulo a total derivative

$$Q^2 = 0 + (\omega^+\varepsilon^{+\mu} + \omega^-\varepsilon^{-\mu})\partial_\mu + \text{equations of motion.} \qquad (4.88)$$

Introducing then a set of antifields Φ^*_i coupled to the nonlinear transformations of the fields $Q\Phi_i$, for the external action we obtain

$$S_{\text{ext}} = \text{Tr} \int \mathrm{d}^4x \left(\Phi^*_i Q\Phi_i + 4g^2\varepsilon^{+\mu}\varepsilon^{-\nu}\left(\varepsilon_{\mu\nu\rho\lambda}A^{*\rho}H^{*\lambda} - \tfrac{1}{2}\left(B^{*\delta}_\nu H^*_{\mu\delta} - B^{*\delta}_\mu H^*_{\nu\delta}\right)\right)\right.$$

$$\left. -\varepsilon_{\mu\nu\rho\lambda}\psi^{*\rho}\chi^{*\lambda} + \tfrac{1}{2}\left(\psi^{*\delta}_\nu \chi^*_{\mu\delta} - \psi^{*\delta}_\mu \chi^*_{\nu\delta}\right)\right)) \qquad (4.89)$$

where, for a p-tensor field

$$\Phi^*_i Q\Phi_i = \frac{1}{p!}\Phi^{*\mu_1...\mu_p}_i Q\Phi_{i\,\mu_1...\mu_p}.$$

Following [29], the gauge-fixing term in the Landau gauge is given by

$$S_{\text{gf}} = Q\ \text{Tr} \int \mathrm{d}^4x \left(\bar{c}\partial^\mu A_\mu\right) + 4g^2\varepsilon^{+\mu}\varepsilon^{-\nu}\ \text{Tr} \int \mathrm{d}^4x\ \varepsilon_{\mu\nu\rho\lambda}\partial^\rho\bar{c}H^{*\lambda} \qquad (4.90)$$

where the antighost $\bar{c}$, introduced by shifting the antifield A_μ^* as $A_\mu^* \to A_\mu^* + \partial_\mu \bar{c}$, is required to transform as

$$Q\bar{c} = b$$
$$Qb = (\omega^+ \varepsilon^{+\mu} + \omega^- \varepsilon^{-\mu}) \partial_\mu \bar{c} \tag{4.91}$$

where b is the Lagrange multiplier. Finally, the complete gauge-fixed action Σ

$$\Sigma = S^{N=4} + S_{\text{ext}} + S_{\text{gf}} \tag{4.92}$$

turns out to obey the following Slavnov–Taylor identity:

$$\mathcal{S}(\Sigma) = \left(\omega^+ \varepsilon^{+\mu} + \omega^- \varepsilon^{-\mu}\right) \Delta_\mu^{\text{cl}} \tag{4.93}$$

with

$$\mathcal{S}(\Sigma) = \text{Tr} \int d^4 x \left(\frac{\delta\Sigma}{\delta\Phi_i^*} \frac{\delta\Sigma}{\delta\Phi_i} + b \frac{\delta\Sigma}{\delta\bar{c}} + \left((\omega^+ \varepsilon^{+\mu} + \omega^- \varepsilon^{-\mu}) \partial_\mu \bar{c}\right) \frac{\delta\Sigma}{\delta b} \right) \tag{4.94}$$

and

$$\Delta_\rho^{\text{cl}} = \text{Tr} \int d^4 x \left(- A_\mu^* \partial_\rho A^\mu - H^{*\mu} \partial_\rho H_\mu - \tfrac{1}{2} B^{*\mu\nu} \partial_\rho B_{\mu\nu} - \tau^* \partial_\rho \tau \right.$$
$$- \tfrac{1}{2} H^{*\mu\nu} \partial_\rho H_{\mu\nu} + \tfrac{1}{2} \psi^{*\mu\nu} \partial_\rho \psi_{\mu\nu} + \tfrac{1}{2} \chi^{*\mu\nu} \partial_\rho \chi_{\mu\nu} + \psi^{*\mu} \partial_\rho \psi_\mu$$
$$\left. + \chi^{*\mu} \partial_\rho \chi_\mu + \xi^* \partial_\rho \xi + \eta^* \partial_\rho \eta - \phi^* \partial_\rho \phi - \overline{\phi}^* \partial_\rho \overline{\phi} + c^* \partial_\rho c \right). \tag{4.95}$$

As before, Δ_ρ^{cl} is linear in the quantum fields, representing a classical breaking not affected by the quantum corrections. The linearized Slavnov–Taylor operator B_Σ

$$B_\Sigma = \text{Tr} \int d^4 x \left(\frac{\delta\Sigma}{\delta\Phi_i^*} \frac{\delta}{\delta\Phi_i} + \frac{\delta\Sigma}{\delta\Phi_i} \frac{\delta}{\delta\Phi_i^*} + b \frac{\delta}{\delta\bar{c}} + (\omega^+ \varepsilon^{+\mu} + \omega^- \varepsilon^{-\mu}) \partial_\mu \bar{c} \frac{\delta}{\delta b} \right) \tag{4.96}$$

is nilpotent modulo a total derivative

$$B_\Sigma B_\Sigma = (\omega^+ \varepsilon^{+\mu} + \omega^- \varepsilon^{-\mu}) \partial_\mu. \tag{4.97}$$

Repeating the same steps as in $N = 2$ SYM, it turns out that the one-to-one relationship (4.76) generalizes [29] to

$$g \frac{\partial\Sigma}{\partial g} = -\frac{\varepsilon^{\mu\nu\rho\sigma}}{96g^2} \mathcal{W}_\mu \mathcal{W}_\nu \mathcal{W}_\rho \mathcal{W}_\sigma \text{Tr} \int d^4 x \, (\omega^{+2} \phi - \omega^{-2} \overline{\phi} + \omega^+ \omega^- \tau)^2 + B_\Sigma \Xi^{-1} \tag{4.98}$$

for some local polynomial Ξ^{-1}. In the present case the climbing-up operator $\mathcal{W}_\mu$ is defined as

$$\mathcal{W}_\mu = \frac{1}{2} \left[\left(\frac{1}{\omega^+} \frac{\partial}{\partial\varepsilon^{+\mu}} + \frac{1}{\omega^-} \frac{\partial}{\partial\varepsilon^{-\mu}} \right), B_\Sigma \right] \tag{4.99}$$

and obeys the same relations (4.73).

Of course, the proof given in [30] can be repeated straightforwardly to show that the insertion $[\text{Tr}(\omega^{+2} \phi - \omega^{-2} \overline{\phi} + \omega^+ \omega^- \tau)^2 \cdot \Gamma]$ has indeed vanishing anomalous dimension. Therefore, according to our theorem, the ultraviolet finiteness of $N = 4$ to all orders of perturbation theory follows from the vanishing of the one-loop beta function β_g, which has been very well known for a long time [19].

5. Conclusion

In this work an algebraic criterion for the ultraviolet finiteness has been presented. The whole framework relies on the analysis of the descent equations following from the integrated consistency condition for invariant counterterms. In some cases, these equations allow one to put in one-to-one correspondence the quantized action with a gauge invariant local field polynomial. The vanishing at the quantum level of the anomalous dimension of this polynomial leads to the finiteness theorem proven in section 3, stating that if the one-loop order coefficient $\beta_g^{(1)}$ vanishes, then β_g vanishes to all orders. The knowledge of the one-loop order beta function $\beta_g^{(1)}$ enables us then to establish whether a given model can be made ultraviolet finite to all orders of perturbation theory. In general, the vanishing of $\beta_g^{(1)}$ can be achieved by an appropriate tuning of the various terms contributing to $\beta_g^{(1)}$, amounting to a suitable choice of the group representations of the field content of the model.

This result shares great analogy with the Adler–Bardeen nonrenormalization theorem for the gauge anomaly. As is well known, the requirement of the vanishing of the one-loop order coefficient of the gauge anomaly results, in fact, in a careful choice for the spinor representations, leading to classify the so-called anomaly-free representations.

We also point out that the present algebraic set up has allowed us to cover the case in which the beta function β_g receives at most one-loop contributions, as in the $N = 2$ SYM.

Finally, it is worth mentioning that although the finiteness theorem has been discussed for models with a single coupling constant, it can be generalized to the case when several couplings are present. Of course, the derivative of the action Σ with respect to each coupling will define a nontrivial element of the integrated cohomology of the linearized Slavnov–Taylor operator B_Σ. The beta functions of those couplings which can be put in correspondence with unrenormalized local polynomials belonging to the cohomology of B_Σ in the lowest level of the descent equations will obey the finiteness theorem. On the other hand, the beta functions of couplings related to nonintegrated cohomology classes in the first level of the descent equations, corresponding to nontrivial pointwise invariant Lagrangians[8], are free to receive quantum corrections.

Acknowledgments

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[8] This is the case, for instance, of the pure Yang–Mills Lagrangian $F_{\mu\nu}(x)F^{\mu\nu}(x)$ which is pointwise invariant under the gauge transformations.

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